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# Some Properties of Faithful Locally Multiplication Modules

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## Abstract

An $R-$module $M$ is called a multiplication module if each submodule of $M$ is a product of an ideal of $R$ by the module and it is called a faithful module if $ann(M)=0$. $M$ is called a locally multiplication module if $M_P$ is a multiplication module for each prime ideal $P$ of $R$ (due to the local ring $R_P$ on which the module $M_P$ is defined). In this paper we explore some new properties of faithful locally multiplication modules.

## Introduction.

Throughout this paper, $R$ is a commutative ring with identity and $M$ is an $R-$module, unless otherwise stated. Let $\emptyset \neq S \subseteq R$, then $S$ is called a multiplicative system if $0 \notin S$ and $a, b \in S$ implies that $ab \in S$ [12]. If $S$ is a multiplicative system in $R$, then the localization of $R$ at $S$ is denoted by $R_S$ (or $S^{-1}R$ [12]), which is the ring $R_S = \{\frac{r}{s}: r \in R, s \in S\}$ [12] and it is called the localization of $R$ at $S$. If $P$ is a prime ideal of $R$, then one can easily get that $R \setminus P$ is a multiplicative system in $R$. In this case, the localization of $R$ at $R \setminus P$ is denoted by $R_P$, so that $R_P = \{\frac{a}{p}: a \in A, p \notin P\}$ and note that $R_P$ is a local ring with $P_P$ as its unique maximal ideal [12]. Now, for $(x,s),(y,t) \in M \times S$, define $(x,s) \sim (y,t)$ if and only if there exists $v \in S$ such that $v(sy - tx) = 0$. This relation is an equivalence relation on $M \times S$ [12]. The equivalence class of an element $(x,s) \in M \times S$ is denoted by $\frac{x}{s}$, so that $\frac{x}{s} = \{(y,t): (y,t) \sim (x,s)\}$ and the set of all such equivalence classes is denoted by $S^{-1}M$ (or $M_S$). That is, $M_S = \{\frac{x}{s}: (x,s) \in M \times S\}$. On $M_S$, define: $\frac{x}{s} + \frac{y}{t} = \frac{tx+sy}{st}$ and $\frac{r}{u}\frac{x}{s} = \frac{rx}{us}$, for all $\frac{r}{u} \in R_S, \frac{x}{s}, \frac{y}{t} \in M_S$. It can be shown that $M_S$ forms an $R_S-$module under these two operations and it is known as the quotient module, or a module of quotients [12]. For a prime ideal $P$ of $R$, the $R_P-$module $M_P$ is called the localization of $M$ at $P$ (due to the local ring $R_P$ on which the module $M_P$ is defined). If $N$ is a submodule of $M$, then $(N:M) = \{r \in R: rM \subseteq N\}$ is an ideal in $R$ [3], in particular, if $N = 0$, then $(0:M) = \{r \in R: rM = 0\}$ is an ideal of $R$, which is known as the annihilator of $M$, $ann(M)$ [3] and $M$ is called a faithful module if $ann(M) = 0$ ***[3]***. In particular, if $N$ is a submodule of $M$, then the annihilator of $N$ is $ann(N) = (0:N) = \{r \in R: rN = 0\}$. We consider $R$ as an $R-$module and $A, B$ are ideals of $R$, then $(A:R) = \{r \in R: rR \subseteq A\}$ and $(A:B) = \{r \in R: rB \subseteq A\}$ which are ideals of $R$. $M$ is called a multiplication $R-$module if for each submodule $N$ of $M$, there exists an ideal $A$ of $R$ such that $N = AM$ [2] and $M$ is called locally multiplication module if $M_P$ is a multiplication module for each prime ideal $P$ of $R$ [6]. $M$ is called finitely generated if $M$ contains a finite generating set, that is there exists a subset $S = \{x_1, x_2, \dots, x_n\} \subseteq M$ such that $M = Rx_1 + Rx_2 + \cdots + Rx_n$, for some positive integer $n$ ***[15]***. If $N$ is a submodule of $M$, then $S_M(N) = \{r \in R: rx \in N,$ for some $x \in M \setminus N\}$ [1]. In particular, if $N = 0$, then $S_M(0) = \{r \in R: rx = 0,$ for some $0 \neq x \in M\}$ and if $A$ is an ideal of $R$, then $S_R(A) = \{r \in R: ra \in A,$ for some $a \notin A\}$ [5]. In particular, if $A = 0$, then $S_R(0) = \{r \in R: ra = 0,$ for some $0 \neq a \in R\}$. $N$ is called a prime submodule of $M$, if $r \in R, m \in M$ such that $rm \in N$, then $m \in N$ or $rM \subseteq N$ [11]. The prime radical $Rad(R)$ of $R$ is defined as $Rad(R) = \bigcap_P P$, where $P$ is a prime ideal of $R$ [4].

## Some Fundamentals.

In this section we recall some results that will be used in the next sections. The proof of which can be found in [10].

**(1)** An $R$ −module $M$ is a multiplication module if and only if $N = (N: M)M$ for every submodule $N$ of $M$.

**(2)** If $R$ is an integral domain and $S$ is a multiplicative system in $R$, then $R_S$ is an integral domain.

**(3)** A multiplication $R$ −module $M$ is locally multiplication.

**(4) [10, Lemma 2.1]**. Let $M$ be an $R$ −module.

(i) Let $N$ be a submodule of $M$ and $P$ is a maximal ideal of $R$ such that $S_M(N) \subseteq P$. If $\frac{x}{p} \in N_P$, where $x \in M$ and $p \notin P$, then $x \in N$. In particular, if $S_M(0) \subseteq P$ and $\frac{x}{p} = 0$ then $x = 0$ and if $N_P = 0$, then $N = 0$.

(ii) Let $I$ be an ideal of $R$ and $P$ is a maximal ideal of $R$ such that $S_R(I) \subseteq P$. If $\frac{r}{q} \in I_P$, where $r \in I$ and $q \notin P$, then $r \in I$. In particular, if $S_R(0) \subseteq P$ and $\frac{r}{p} = 0$ then $r = 0$ and if $I_P = 0$, then $I = 0$.

**(5) [9, Proposition 2.1]**. Let $L$ and $N$ be submodules of an $R$ −module $M$. Then $L \subseteq N$ if an only if $L_P \subseteq N_P$ for every maximal ideal $P$ of $R$.

**(6) [9, Corollary 2.9]**. Let $M$ be an $R$ −module and $P$ be a prime ideal (resp. a maximal ideal) of $R$. If $r \in R$, $p \notin P$ and $x \in M$, then $\frac{r}{p}M_P = (rM)_P$ and $(Rx)_P = R_P\frac{x}{p}$. In particular, $\frac{r}{1}M_P = (rM)_P$ and $(Rx)_P = R_P\frac{x}{1}$.

**(7) [7, Lemma 4.8]**. Let $M$ be an $R$ −module and $P$ a prime ideal of $R$. Let $N, L$ be proper submodules of $M$. Then, the following hold.

(i) If $L_P \subseteq N_P$ and $S_M(N) \subseteq P$, then $L \subseteq N$.

(ii) If $N_P = L_P$ and $S_M(N), S_M(L) \subseteq P$, then $N = L$.

**(8) [8, Theorem 2.2]**. Let $M$ be an $R$ −module and $S$ be a multiplicative system in $R$. Let $\overline{N}$ be a submodule of $M_S$, then $\overline{N} = N_S$ for some submodule $N$ of $M$ with $S_M(N) \cap S = \emptyset$. Furthermore, if $\overline{N}$ is proper, then $N$ is proper. In particular $\overline{N} = N_P$ with $S_M(N) \subseteq P$.

**(9) [13, Proposition 2.1]**. Let $R$ be an integral domain. If $M$ is a faith- ful multiplication $R$ −module, then every non-zero submodule of $M$ is faithful.

**(10) [14, Theorem 9]**. Let $M$ be a multiplication $R$ −module with annihilator $J$ and $A$ and $B$ ideals of $R$. Then $AM \subseteq BM$ if and only if $A \subseteq B + J$ or $M = \big((B + J): A\big)M$.

## Some Preliminary Results

In this section we prove some preliminary results which will be used in proving some properties of faithful locally multiplication modules in the next section. In the first result of this section we prove that the faithfulness property is an equivalent property between those modules over integral domains and their localizations.

Recall that a submodule $N$ od an $R-$module $M$ is called a faithful submodule if $ann(N) = 0$.

**Proposition 3.1**. If $M$ is an $R-$module, where $R$ is an integral domain and $N$ a submodule of $M$, then $N$ is faithful if and only if $N_P$ is faithful, for every prime ideal $P$ of $R$

**Proof**. Let $N$ be faithful, that is, $ann(N) = 0$, and $P$ any prime ideal of . To show that $N_P$ is a faithful $R_P-$module, let $\frac{m}{p} \in ann(N_P)$. Then $\frac{m}{p} N_P = 0$. That means , $(mN)_P = 0$. It follows that there exists $q \notin P$ such that $q(mN) = 0$, so that we get $(qm)N = 0$, this means $qm \in ann(N) = 0$. Thus $qm = 0$. Since, $R$ is an integral domain and $q \neq 0$ (since $q = 0$ implies that $q \in P$), so that $m = 0$, thus $\frac{m}{p} = 0$. This gives that $ann(N_P) = 0$. Hence, $N_P$ is a faithful $R_P-$module.

Conversely, suppose that $N_P$ is faithful for every prime ideal $P$ of $R$. As $R$ is a commutative ring with identity, it contains at least a maximal ideal $P$ of . Since $N_P$ is a faithful $R_P-$module, $ann(N_P) = 0$. To show that $N$ is faithful, let $m \in ann(N)$. Then $mN = 0$. From which we get $\frac{m}{1} N_P = (mN)_P = 0$. Hence, $\frac{m}{1} \in ann(N_P) = 0$, so that $pm = 0$ for some $p \notin P$ and as $R$ is an integral domain and $p \neq 0$ we get $m = 0$. Thus $ann(N) = 0$. Hence, $N$ is a faithful $R-$module.

In the next result we prove that the localization of residual of submodules is contained in the residual of the localizations of the submodules.

**Proposition 3.2**. Let $M$ be an $R-$module and $N, L$ submodules of $M$. If $P$ is a prime ideal of $R$, then $(N:L)_P \subseteq (N_P:L_P)$.

**Proof**. Let $\frac{a}{p} \in (N:L)_P$, for $a \in R$ and $p \notin P$, then $qa \in (N:L)$, for some $q \notin P$. Consequently, $qaL \subseteq N$. If $\frac{m}{t} \in L_P$ is any element, where $m \in M$ and $t \notin P$, then $sm \in L$ for some $s \notin P$. This gives that $qasm \in N$. Now, we have $\frac{a}{p}.\frac{m}{t} = \frac{q}{q}.\frac{s}{s}.\frac{am}{pt} = \frac{qasm}{qspt} \in N_P$, so that $\frac{a}{p} L_P \subseteq N_P$, that means $\frac{a}{p} \in (N_P:L_P)$. Hence, $(N:L)_P \subseteq (N_P:L_P)$.

Now we give a condition under which the residual of the localizations of submodules is contained in the localization of the residual of the submodules.

**Proposition 3.3.** Let $M$ be an $R-$module and $N, L$ submodules of $M$. If $P$ is any prime ideal of $R$ with $S_M(N) \subseteq P$, then $(N_P:L_P) \subseteq (N:L)_P$.

**Proof.** Suppose that $\frac{a}{p} \in (N_P:L_P)$, then $\frac{a}{p} L_P \subseteq N_P$. To show $aL \subseteq N$. Let $l \in L$ be any element. Then $\frac{l}{1} \in L_P$. Thus $\frac{a}{p}.\frac{l}{1} \in N_P$, so $sal \in N$, for some $s \notin P$. As $s \notin S_M(N)$, we get $al \in N$. Hence $aL \subseteq N$, that means, $a \in (N:L)$, so we get $\frac{a}{p} \in (N:L)_P$, which gives that $(N_P:L_P) \subseteq (N:L)_P$..

Now, by combining **Proposition 3.2** and **Proposition 3.3**, we give the following result.

**Theorem 3.4** Let $M$ be an $R-$module and $N, L$ a proper submodule of $M$. If $P$ is any prime ideal of $R$ with $S_M(N) \subseteq P$, then $(N:L)_P = (N_P:L_P)$..

**Proof**. The proof follows directly from **Proposition 3.2** and **Proposition 3.3**.

If we consider $R$ as an $R$ −module, then we give the following corollary from **Theorem 3.4**.

**Corollary 3.5**. Let $A$ and $B$ ideals of $R$. If $P$ is a prime ideal of $R$ such that $S_R(A) \subseteq P$, then $(A{:}\,B)_P = (A_P{:}\,B_P)$.

**Proof**. By considering $R$ as an $R$ −module, the proof follows directly by **Theorem 3.4**.

Now we prove the following result..

**Theorem 3.6**. Let $M$ be an $R$ −module and let $N$ and $L$ be submodules of $M$. Then $N \subseteq L$ if and only if $N_P \subseteq L_P$, for every maximal ideal $P$ of $R$.

**Proof**. Let $N \subseteq L$ and $P$ be any maximal ideal of  . To show that $N_P \subseteq L_P$, let $\frac{x}{p} \in N_P$, where $x \in M$ and $p \notin P$. Then $\frac{x}{p} = \frac{n}{q}$, for some $n \in N$ and $q \notin P$. It follows that there exists $t \notin P$ such that $t(qx - pn) = 0$. This gives that $tqx = tpn \in N \subseteq L$, Then we have $\frac{x}{p} = \frac{t\,q\,x}{t\,q\,p} = \frac{tqx}{tqp} \in L_P$. Hence, $N_P \subseteq L_P$.

Conversely, suppose that $N_P \subseteq L_P$, for every maximal ideal $P$ of $R$ and to show that $N \subseteq L$. Let $x \in N$, then $\frac{x}{1} \in N_P \subseteq L_P$, for every maximal ideal $P$ of $R$. Hence, for every maximal ideal $P$ of $R$, there exists an element $r_P \notin P$ such that $r_P x \in L$. Let $A$ be the ideal of $R$ generated by $r_P$', that is, $A =< S >$, where $S = \{r_P{:}\, P$ is a maximal ideal of $R\}$. If $A \neq R$, then $A$ contained in some maximal ideal $Q$ of $R$, this implies that $r_Q \in Q$, which is a contradiction, since on the other hand the element $r_Q$ of $S$ is chosen in such a way such that $r_Q \notin Q$. Therefore $A = R$. Obviously, $1 \in A =< S >$, so that there exists a finite number of maximal ideals $P_1, P_2, \dots, P_n$ of $R$ with $r_{P_i} \notin P_i$, for $i$ $(1 \leq i \leq n)$, such that $1 = r_1 r_{P_i} + r_2 r_{P_2} + \cdots + r_n r_{P_n}$, where $r_1, r_2, \dots, r_n \in R$. Now, we have $x = 1.x = \left(r_1 r_{P_i} + r_2 r_{P_2} + \cdots + r_n r_{P_n}\right).x = r_1 r_{P_i} x + r_2 r_{P_2} x + \cdots + r_n r_{P_n} x \in L$ . Thus $N \subseteq L$.

Next, we give the following corollary which determines the relation between equality of submodules and equality of their localizations at maximal ideals.

**Corollary 3.7**. Let $M$ be an $R$ −module and let $N$ and $L$ be submodules of $M$. Then $N = L$ if and only if $N_P = L_P$, for every maximal ideal $P$ of $R$.

**Proof**. The proof follows from **Theorem 3.6**.

It is known that if $A, B$ are ideals of $R$ such that $A \subseteq B$ and $S$ is a multiplicative system in $R$ ($P$ is a prime ideal of $R$), then $A_S \subseteq B_S$ $(A_P \subseteq B_P)$. Now we give a condition under which the converse of the above is also true.

**Lemma 3.8**. Let $R$ be a commutative ring with identity and $S$ a multiplicative system in $R$ ($P$ is a prime ideal of $R$) and $A, B$ are ideals of $R$. If $S_R(B) \cap S = \emptyset$ $(S_R(B) \subseteq P)$ and $A_S \subseteq B_S$ $(A_P \subseteq B_P)$, then $A \subseteq B$.

**Proof**. We prove the first part. Let $a \in A$. Take $s \in S$ (this is possible since $S \neq \emptyset$), then $\frac{a}{s} \in A_S \subseteq B_S$, so that $pa \in B$ for some $p \in S$. If $a \notin B$, then we get $p \in S_R(B)$, so that we get $S_R(B) \cap S \neq \emptyset$ which is a contradiction and thus $a \in B$. Hence, we get $A \subseteq B$. For the second part, let $a \in A$, then $\frac{a}{1} \in A_P \subseteq B_P$, so that $pa \in B$ for some $p \notin P$. If $a \notin B$, then we get $p \in S_R(B) \subseteq P$ which is a contradiction and thus $a \in B$. Hence, we get $A \subseteq B$. The proof of the particular case is obvious by taking $A = R$.

Next, we prove that the localization of annihilator of a submodule is contained in the annihilator of the localization of the submodule.

**Proposition 3.9**. Let $M$ be an $R$ −module and $N$ a submodule of $M$. If $P$ is a prime ideal of $R$, then $(ann(N))_P \subseteq ann(N_P)$..

**Proof**. If $\frac{r}{p} \in (ann(N))_P$, for $r \in R$ and $p \notin P$, then $qr \in ann(N)$, for some $q \notin P$, so that $qrN = 0$. Let $\frac{m}{u} \in N_P$ , where $m \in M$ and $u \notin P$. Then we have $vm \in N$, for some $v \notin P$, so that $qrvm = 0$. Thus $\frac{r}{p}\frac{m}{u} = \frac{q}{q}\frac{r}{p}\frac{v}{v}\frac{m}{u} = \frac{qrvm}{qpvu} = 0$ and $\frac{r}{p} \in ann(N_P)$, so $(ann(N))_P \subseteq ann(N_P)$.

Now we give a condition which makes the localization of annihilator of a submodule and the annihilator of the localization of the submodule are equal.

**Proposition 3.10**. Let $M$ be an $R$ −module and $N$ a submodule of $M$. If $P$ is a prime ideal of $R$ with $S_M(0) \subseteq P$, then $ann(N_P) = (ann(N))_P$.

**Proof**. In view of **Proposition 3.9**, it is enough to show that $ann(N_P) \subseteq (ann(N))_P$. Now,

let $\frac{r}{p} \in ann(N_P)$, for $r \in R, p \notin P$. If $m \in N$, then $\frac{m}{1} \in N_P$ . Thus $\frac{rm}{p} = \frac{r}{p}\frac{m}{1} = 0$, so by **[10, Lemma 2.1]** , we get $rm = 0$ . Thus $r \in ann(N)$ consequently, $\frac{r}{p} \in (ann(N))_P$. Thus, $ann(N_P) \subseteq (ann(N))_P$. Hence, $ann(N_P) = (ann(N))_P$.

In the following proposition we prove that for an $R$ −module $M$ and a prime ideal $P$ of $R$, each prime submodule of $M_P$ is a localization of some submodule of $M$.

**Proposition 3.11**. Let $M$ be an $R$ −module and $P$ a prime ideal of $R$. If $\overline{N}$ is a prime submodule of $M_P$, then $\overline{N} = N_P$, where $N = \{x \in M : \frac{x}{1} \in \overline{N}\}$ is a prime submodule of $M$ and $S_M(N) \subseteq P$.

**Proof**. It is easy to show that $N$ is a submodule of $M$. To show $\overline{N} = N_P$. Let $\frac{x}{p} \in \overline{N}$, where $x \in M$ and $p \notin P$. Now, we have $\frac{x}{1} = \frac{p}{p}.\frac{x}{1} = \frac{p}{1}.\frac{x}{p} \in \overline{N}$, thus we get $x \in N$ and hence $\frac{x}{p} \in N_P$, which means that $\overline{N} \subseteq N_P$. Next, let $\frac{x}{p} \in N_P$, where $x \in M$ and $p \notin P$. Then we have $qx \in N$, for some $q \notin P$ and then we get $\frac{qx}{1} \in \overline{N}$, so that $\frac{x}{p} = \frac{q}{q}.\frac{x}{p} = \frac{1}{qp}.\frac{qx}{1} \in \overline{N}$. That means $N_P \subseteq \overline{N}$. Hence $\overline{N} = N_P$. It remains to show that $N$ is a prime submodule of $M$. If $N = M$, then $\overline{N} = N_P = M_P$, which is a contradiction. Therefore $N$ is a proper submodule of $M$. Let $rx \in N$, for $r \in R$ and $x \in M$. Then $\frac{rx}{1} \in \overline{N} = N_P$, that means $\frac{r}{1}.\frac{x}{1} \in \overline{N}$ and as $\overline{N}$ is prime we have either $\frac{x}{1} \in \overline{N}$ or $\frac{r}{1}M_P \subseteq N_P = \overline{N}$. Then either $x \in N$ or $(rM)_P = \frac{r}{1}M_P \subseteq N_P = \overline{N}$. Now, if $m \in M$, then $\frac{rm}{1} \in (rM)_P \subseteq \overline{N}$. That means $rm \in N$ for all $m \in M$ and thus $rM \subseteq N$. Hence $N$ is a prime submodule of $M$. It remains to show that $S_M(N) \subseteq P$. Let $r \in S_M(N)$. Ten $rx \in N$, for some $x \notin N$. Thus $\frac{rx}{1} \in \overline{N}$. If $r \notin P$, then $\frac{x}{1} = \frac{r}{r}\frac{x}{1} = \frac{1}{r}\frac{rx}{1} \in \overline{N}$, so that $x \in N$, that is a contradiction. Hence, $r \in P$, so $S_M(N) \subseteq P$.

Here, we prove that each prime ideal of $R_P$ is a localization of some prime ideal of $R$.

**Proposition 3.12**. Let $P$ be a prime ideal of $R$. If $\bar{A}$ is a prime ideal of $R_P$, then $\bar{A} = A_P$, where $A = \{x \in R : \frac{x}{1} \in \bar{A}\}$ is a prime ideal of $R$ and $S_R(A) \subseteq P$.

**Proof**. It is easy to show that $A$ is an ideal of $R$. To show that $\bar{A} = A_P$, let $\frac{x}{p} \in \bar{A}$, where $x \in R$ and $p \notin P$. Now, we have $\frac{x}{1} = \frac{p}{p}.\frac{x}{1} = \frac{p}{1}.\frac{x}{p} \in \bar{A}$. Thus we get $x \in A$ and hence $\frac{x}{p} \in A_P$, which means that $\bar{A} \subseteq A_P$. Next, let $\frac{x}{p} \in A_P$, where $x \in R$ and $p \notin P$. Then we have $qx \in A$, for some $q \notin P$ and then we get $\frac{qx}{1} \in \bar{A}$, so that $\frac{x}{p} = \frac{q}{q}.\frac{x}{p} = \frac{1}{qp}.\frac{qx}{1} \in \bar{A}$. That means $A_P \subseteq \bar{A}$. Hence $\bar{A} = A_P$. Finally we will show that $A$ is a prime ideal of $R$. If $A = R$, then $\bar{A} = A_P = R_P$, which is a contradiction, so that $A$ is a proper ideal of $R$. Let $rx \in A$, for $r, x \in R$. Then $\frac{rx}{1} \in \bar{A} = A_P$, that means $\frac{r}{1}.\frac{x}{1} \in \bar{A}$ and as $\bar{A}$ is prime we have either $\frac{r}{1} \in \bar{A}$ or $\frac{x}{1} \in \bar{A}$. It follows that $r \in A$ or $x \in A$. Hence, $A$ is a prime ideal of $R$. It remains to show that $S_R(A) \subseteq P$. Let $r \in S_R(A)$, then $rx \in A$, for some $x \notin A$. Thus $\frac{rx}{1} \in \bar{A}$. If $r \notin P$, then $\frac{x}{1} = \frac{r}{r}\frac{x}{1} = \frac{1}{r}\frac{rx}{1} \in \bar{A}$, so that $x \in A$, that is a contradiction. Hence, $r \in P$, so $S_R(A) \subseteq P$.

In the following result we give a condition under which the primness property of ideals can be extended from the ring to its localization .

**Proposition 3.13**. Let $P$ be a prime ideal of $R$. If $A$ is an ideal of $R$ such that $S_R(A) \subseteq P$, then $A$ is a prime ideal of $R$ if and only if $A_P$ is a prime

ideal of $R_P$.

**Proof**. Let $A$ be a prime ideal of $R$. If $A_P = R_P$, then $\frac{1}{1} \in A_P$. Thus $p.1 \in A$, for some $\notin P$ . Since, $A$ is proper, so $1 \notin A$. Hence, $p \in S_R(A) \subseteq P$, which is impossible. Hence, $A_P$ is a proper ideal of $R_P$. Next, let $a, b \in R$ and $p, q \notin P$ be such that $\frac{a}{p}.\frac{b}{q} \in A_P$. Then $\frac{ab}{pq} \in A_P$. Thus $sab \in A$ for some $s \notin P$. If $s \in A$, then $s.1 = s \in A$, but $1 \notin A$, so that $s \in S_R(A) \subseteq P$, which is a contradiction, so that $s \notin A$ . Since $A$ is prime, either $a \in A$ or $b \in A$, that is $\frac{a}{p} \in A_P$ or $\frac{b}{q} \in A_P$. Hence, $A_P$ is a prime ideal of $R_P$.

Conversely, suppose that $A_P$ is a prime ideal of $R_P$. If $A = R$, then $A_P = R_P$, which contradicts the fact that $A_P$ is proper in $R_P$), that is $A \neq R$.. Now assume $ab \in A$. Then $\frac{a}{1}.\frac{b}{1} = \frac{ab}{1} \in A_P$. Since $A_P$ is prime, we get $\frac{a}{1} \in A_P$ or $\frac{b}{1} \in A_P$. If $\frac{a}{1} \in A_P$, then $pa \in A$ for some $p \notin P$. If $a \notin A$, then $p \in S_R(A) \subseteq P$, which is a contradiction, so that $a \in A$. If $\frac{b}{1} \in A_P$, then by the same technique, we get $b \in A$. Hence, $A$ is a prime ideal of $R$.

Next we prove the following lemma.

**Lemma 3.14.** If $N$ is a submodule of an $R-$module $M$ and $P$ is a prime ideal of $R$ such that

$S_M(N) \subseteq P$, then $N$ is a proper submodule of $M$ if and only if $N_P$ is a proper submodule of

$M_P$.

**Proof.** Suppose that $N \neq M$. Then there exists $x \in M$ but $x \notin N$ such that $\frac{x}{1} \in M_P$. If $\frac{x}{1} \in N_P$, then $px \in N$, for some $p \notin P$ and then we get $p \in S_M(N) \subseteq P$, which is a contradiction. Therefore $\frac{x}{1} \notin N_P$. Thus $N_P \neq M_P$. Conversely, suppose that $N_P \neq M_P$. Then there exists $x \in M$ and $p \notin P$ such that $\frac{x}{p} \in M_P$ with $\frac{x}{p} \notin N_P$. This last assertion implies that $x \notin N$ (otherwise, $\frac{x}{p} \in N_P$). Hence, $N \neq M$.

In the following result we give a condition under which the primness property of submodules can be extended from the module to its localization and conversely.

**Proposition 3.15**. Let $M$ be an $R$ −module and $P$ a prime ideal of $R$. If $N$ is a submodule of $M$ such that $S_M(N) \subseteq P$, then $N$ is a prime submodule of $M$ if and only if $N_P$ is a prime submodule of $M_P$.

**Proof**. Suppose that $N$ is a prime submodule of $M$, so $N$ is a proper submodule of $M$. Hence, by **Lemma 3.14**, we have $N_P$ is a proper submodule of $M_P$. Let $\frac{r}{p}\frac{m}{q} \in N_P$, for $r \in R, m \in M$ and $p, q \notin P$. Then $\frac{rm}{pq} \in N_P$ and so by **[10, Lemma 2.1]**, we have $rm \in N$. Since, $N$ is a prime submodule of $M$, either $m \in N$ or $rM \subseteq N$. If $m \in N$, then $\frac{m}{q} \in N_P$. If $rM \subseteq N$, then, by using **[9, Proposition 2.1]** and **[9, Corollary 2.9]**, we get $\frac{r}{p}M_P = (rM)_P \subseteq N_P$. Hence $N_P$ is a prime submodule of $M_P$.

Conversely, suppose that $N_P$ is a prime submodule of $M_P$. If $N = M$, then $N_P = M_P$, which contradicts that $N_P$ is a proper submodule of $M_P$). Thus $N \neq M$. Now, let $rm \in N$, where $r \in R, m \in M$. Then $\frac{r}{1}\frac{m}{1} \in N_P$. As $N_P$ is prime, we have $\frac{m}{1} \in N_P$ or $\frac{r}{1}M_P \subseteq N_P$. If $\frac{m}{1} \in N_P$, then by **[10, Lemma 2.1]**, we get $m \in N$. If $\frac{r}{1}M_P \subseteq N_P$, then **[9, Corollary 2.9]**, we have $(rM)_P = \frac{r}{1}M_P \subseteq N_P$. If $m \in M$ is any element, then $\frac{rm}{1} = \frac{r}{1}.\frac{m}{1} \in \frac{r}{1}M_P \subseteq N_P$, so by **[10, Lemma 2.1]**, we get $rm \in N$. Thus we get $rM \subseteq N$. Hence $N$ is a prime submodule of $M$.

In the next result we give a condition under which the maximality property of submodules can be extended from the module to its localization

**Proposition 3.16**. Let $M$ be an $R$ −module and $P$ be a prime ideal of $R$. If $N$ is a maximal submodule of $M$ such that $S_M(N) \subseteq P$, then $N_P$ is a maximal submodule of $M_P$.

**Proof**. Since, $N$ is a maximal submodule of $M$, $N$ is a proper submodule of $M$. Hence, by **Lemma 3.14**, we have $N_P$ is a proper submodule of $M_P$. Let $N_P \subseteq \overline{L} \subseteq M_P$, where $\overline{L}$ is a submodule of $M_P$. Then, by **[8, Theorem 2.2]**, we get that $\overline{L} = L_P$, for some submodule $L$ of $M$ with $S_M(L) \subseteq P$. Then, $N_P \subseteq L_P \subseteq M_P$, so by **[7, Lemma 4.8]**, we get $N \subseteq L$. Hence, we get $N \subseteq L \subseteq M$. Since $N$ is maximal, either $N = L$ or $L = M$, that is, $N_P = L_P = \overline{L}$ or $\overline{L} = L_P = M_P$. Hence, $N_P$ is a maximal submodule of $M_P$.

In the last two results of this section we prove that for a commutative ring $R$ and a multiplicative system $S$ in $R$ (a prime ideal $P$ of $R$), each ideal in $R_S$ $(R_P)$ is a localization of unique ideal in $R$.

**Proposition 3.17**. Let $S$ be a multiplicative system in $R$. If $\overline{A}$ is an ideal in $R_S$, then there exists a unique ideal $A$ of $R$ with the property that $S_R(A) \cap S = \emptyset$ and $\overline{A} = A_S$.

**Proof**. Let $s \in S$ be any element and $A = \{x \in R : \frac{x}{s} \in \bar{A}\}$.

Since $0 \in R$ and $\frac{0}{s} \in \bar{A}$, so that $0 \in A$. Hence, $\emptyset \neq A \subseteq R$. Now, let $r \in R$ and $x, y \in A$, then $x, y \in R$ and $\frac{x}{s}, \frac{y}{s} \in \bar{A}$, then $\frac{x-y}{s} = \frac{x}{s} - \frac{y}{s} \in \bar{A}$ and $x - y \in A$. Now, we have $\frac{r}{s} \in R_S$. $\frac{rx}{s} = \frac{s}{s}\frac{s}{s}\frac{rx}{s} = \frac{ss}{s}\frac{rx}{ss} \in \bar{A}$, so that $rx \in A$. Since, $R$ is commutative, so $xr = rx \in A$. Hence, $A$ is an ideal of $R$. To show $\bar{A} = A_S$. Let $\frac{x}{t} \in \bar{A}$, where $x \in R, t \in S$. Then $\frac{x}{s} = \frac{t}{t}\frac{x}{s} = \frac{t}{s}\frac{x}{t} \in \bar{A}$. This gives $x \in A$ and $\frac{x}{s} \in A_S$. Hence, $\bar{A} \subseteq A_S$. Next, let $\frac{x}{t} \in A_S$, where $x \in R, t \in S$. Then $qx \in A$, for some $q \in S$, so that $\frac{qx}{s} \in \bar{A}$. Obviously, $\frac{x}{t} = \frac{s}{s}\frac{q}{q}\frac{x}{t} = \frac{s}{tq}\frac{qx}{s} \in \bar{A}$, so that $A_S \subseteq \bar{A}$. Hence, $\bar{A} = A_S$. If $S \cap S_R(A) \neq \emptyset$, then there exists $t \in S \cap S_R(A)$, so that $t \in S$ and $tx \in A$, for some $x \notin A$, that is $\frac{tx}{s} \in \bar{A}$ and $\frac{x}{s} \notin \bar{A}$. On the other hand, we have, $\frac{x}{s} = \frac{t}{t}\frac{x}{s} = \frac{1}{t}\frac{tx}{s} \in \bar{A}$, which is a contradiction. Therefore $S \cap S_R(K) = \emptyset$. Next, suppose that $B$ is another ideal of $R$, for which $\bar{A} = B_S$ with the property that

$S \cap S_R(B) = \emptyset$. To show that, $B = A$. Now, we have, $B_S = A_S$. Let $x \in B$. Then $\frac{x}{s} \in B_S = \bar{A}$, so that $x \in A$. Hence, $B \subseteq A$.

Conversely, let $x \in A$. Then $\frac{x}{s} \in \bar{A} = A_S = B_S$, this implies that $qx \in B$, for some $q \in S$. If $x \notin B$, then $q \in S_R(B)$. Which implies that $S \cap S_R(B) \neq \emptyset$. Therefore $x \in B$ and thus $A \subseteq B$. Hence $B = A = \{x \in R: \frac{x}{s} \in A_S = B_S\}$.

**Corollary 3.18**. Let $P$ be a prime ideal of $R$. If $\overline{A}$ is an ideal of $R_P$, then there exists a unique ideal $A = \{a \in R: \frac{a}{1} \in \overline{A}\}$ of $R$ with the property that $S_R(A) \subseteq P$ and $\overline{A} = A_P$.

**Proof**. Since $P$ is a prime ideal of $R$, $R \backslash P$ is a multiplicative system in $R$ and then by taking $S = R \backslash P$ in **Proposition 3.17**, we have $S_R(A) \subseteq P$ if and only if $S \cap S_R(A) = \emptyset$. If $S_R(A) \subseteq P$, then $S \cap S_R(A) = (R \backslash P) \cap S_R(A) \subseteq (R \backslash P) \cap P = \emptyset$. It follows that $(R \backslash P) \cap S_R(A) = \emptyset$, so that if $r \in S_R(A)$, then $r \notin R \backslash P$. Thus $r \in P$ and hence, we get $S_R(A) \subseteq P$. By Proposition **3.17**, there is a unique ideal $A$ of $R$ with $S_R(A) \subseteq P$, where the ideal $A = \{x \in R: \frac{x}{s} \in \overline{A}\}$, where $s \in S = R \backslash P$. We can take $s = 1 \in S = R \backslash P$. That means $A = \{a \in R: \frac{a}{1} \in \overline{A}\}$ is the required unique ideal of $R$ for which $S_R(A) \subseteq P$ and $\overline{A} = A_P$.

## Some Properties of Faithful Locally Multiplication Modules

This section is devoted to prove some properties of faithful locally multiplication modules.

We start this section with the following result which states that: Nonzero submodules of faithful locally multiplication modules over integral domains are faithful under certain conditions.

**Proposition 4.1**. If $M$ is a faithful locally multiplication $R$ −module, where $R$ is an integral domain and $N$ is a non-zero submodule of $M$ such that $S_M(N) \subseteq rad(R)$, then $N$ is faithful.

**Proof**. Let $N$ be any non-zero submodule of $M$ and $P$ a prime ideal of $R$. As $R$ is an integral domain and $M$ is faithful, by **Proposition 3.1**, we get $M_P$ is a faithful $R_P$ −module. Since $M$ is locally multiplication, we have $M_P$ is a multiplication $R_P$ −module and also we have $S_M(N) \subseteq rad(R) \subseteq P$. Now, $N_P$ is a submodule of $M_P$. If $N_P = 0$, then by **[10, Lemma 2.10]**, we get $N = 0$ which is a contradiction, so that $N_P \neq 0$. Hence, by **[13, Proposition 2.1]**, we get $N_P$ is faithful and as $R$ is an integral domain, by **Proposition 3.1**, we get $N$ is faithful.

**Proposition 4.2**. Let $M$ be a locally multiplication $R$ −module such that $S_M(0) \subseteq Rad(R)$ and let $A, B$ be ideals of $R$ such that $S_R(B + ann(M)) \subseteq Rad(R)$. Then $AM \subseteq BM$ if and only if $A \subseteq B + ann(M)$ or $M = \Big(\big(B + ann(M)\big): A\Big)M$.

**Proof**. Let $AM \subseteq BM$ and $P$ be any maximal ideal of $R$. Then $P$ is prime. Since $S_M(0) \subseteq Rad(R) \subseteq P$, $M_P$ is a multiplication $R_P$ −module and $A_P$, $B_P$ are ideals of $R_P$ such that $A_P M_P = (AM)_P \subseteq (BM)_P = B_P M_P$, so by **[14, Theorem 9]**, we get $A_P \subseteq B_P + ann(M_P)$ or $M_P = ((B_P + ann(M_P)): A_P)M_P$. First, suppose that $M_P = ((B_P + ann(M_P)): A_P)M_P$ for all maximal ideals $P$ of $R$. As, $S_M(0) \subseteq P$, by **Proposition 3.10**, we get $ann(M_P) = (ann(M))_P$, so that we get $B_P + ann(M_P) = B_P + (ann(M))_P = (B + ann(M))_P$ and as $S_R(B + ann(M)) \subseteq Rad(R) \subseteq P$, by **Corollary 3.5**, we get $M_P = \Big(\big(B_P + ann(M_P)\big): A_P\Big)M_P = \Big(\big(B + ann(M)\big)_P: A_P\Big)M_P = ((B + ann(M)): A)_P M_P = (((B + ann(M)): A)M)_P$. From **Corollary 3.7**, we get $M = ((B + ann(M)): A)M$. Next, suppose that for some maximal ideal $P$ of $R$ we have $A_P \subseteq B_P +$

$ann(M_P)$. Then $A_P \subseteq (B + ann(M))_P$ and as $S_R(B + ann(M)) \subseteq P$. From **Lemma 3.8**, we get $A \subseteq B + ann(M)$. Conversely, suppose that $A \subseteq B + ann(M)$ or $M = \left((B + ann(M)) : A\right) M$. Let $P$ be any maximal ideal of $R$, so it is prime and hence, $M_P$ is a multiplication $R_P$ −module. Now, if $A \subseteq B + ann(M)$, then by **Theorem 3.6**, we get $A_P \subseteq \left(B + ann(M)\right)_P = B_P + \left(ann(M)\right)_P = B_P + ann(M_P)$. Since $M_P$ is a multiplication $R_P$ −module, so by **[14, Proposition 9]**, we get $A_P M_P \subseteq B_P M_P$ and then $(AM)_P = A_P M_P \subseteq B_P M_P = (BM)_P$. By **Theorem 3.6**, we get $AM \subseteq BM$. If $M = \left((B + ann(M)) : A\right) M$, then $M_P = \left(((B + ann(M)) : A) M\right)_P = ((B + ann(M)) : A)_P M_P = \left((B + ann(M))_P : A_P\right) M_P = \left((B_P + ann(M_P)) : A_P\right) M_P$. Since $M_P$ is a multiplication $R_P$ −module, so by **[14, Proposition 9]**, we get $A_P M_P \subseteq B_P M_P$ and then we get $(AM)_P = A_P M_P \subseteq B_P M_P = (BM)_P$ and this last result is true for all maximal ideals $P$ of $R$, so by **Theorem 3.6**, we get $AM \subseteq BM$.

It is known that, if $M$ is a multiplication module and $N$ is a submodule of $M$, then $N = AM$ for some ideal $A$ of $R$. In the following result we prove that each prime submodule $N$ of a faithful locally multiplication module $N$ contains $QM$ for some prime ideal $Q$ of $R$ under some conditions.

**Proposition 4.3**. Let $M$ be a locally multiplication $R$ −module. If $N$ is a prime submodule of $M$ such that $S_M(0), S_M(N) \subseteq Rad(R)$, then there exists a prime ideal $Q$ of $R$ such that $ann(M) \subseteq Q$ and $QM \subseteq N$.

**Proof**. As $R$ is commutative with identity, it contains at least one maximal ideal $P$, then $P$ is a prime ideal of $R$, so that $M_P$ is a multiplication $R_P$ −module. Thus $S_M(N) \subseteq Rad(R) \subseteq P$. By **[10, Proposition 2.21]**, we get $N_P$ is a prime submodule of the multiplication $R_P$ −module $M_P$. Hence, by **[13, Theorem 2.3]**, there exists a unique prime ideal $\overline{P}$ of $R_P$ such that $ann(M_P) \subseteq \overline{P}$ and $N_P = \overline{P} M_P$. Now, by **[12, Corollary 3.11]**, there exists a prime ideal $Q$ such that $\overline{P} = Q_P$ and $Q \subseteq P$. As, $S_M(0) \subseteq Rad(R) \subseteq P$, by **Proposition 3.10**, we get $ann(M_P) = (ann(M))_P$. Thus we get $(ann(M))_P \subseteq Q_P$. Let $x \in ann(M)$, then $\frac{x}{1} \in (ann(M))_P \subseteq Q_P$, so that $px \in Q$ for some $p \notin P$. Obviously, $p \notin Q$ and as $Q$ is prime, we get $x \in Q$. Hence, we get $ann(M) \subseteq Q$. Next, to show $QM \subseteq N$. Let $x \in QM$, then $\frac{x}{1} \in (QM)_P = N_P$, so that $qx \in N$ for some $q \notin P$. If $x \notin N$, then $q \in S_M(N) \subseteq P$ which is a contradiction. Hence, $x \in N$, so that $QM \subseteq N$.

In the next two results we give some conditions under which the multiplication of prime ideals (maximal ideals) by faithful locally multiplication modules are prime submodules (maximal submodules).

**Proposition 4.4**. Let $M$ be a faithful locally multiplication $R$ −module and $P$ a prime ideal of $R$ such that $S_M(0), S_M(PM) \subseteq P$. Then, $PM$ is a prime submodule of $M$ if and only if $(PM : M) = P$.

**Proof**. Let $PM$ be a prime submodule of $M$. Since, $M$ is locally multiplication, we get that $M_P$ is a multiplication $R$ −module and since, $S_M(0) \subseteq P$, by **Proposition 3.10**, we get $ann(M_P) = (ann(M))_P$. Since, $M$ is faithful, so that $ann(M) = 0$, and hence, $ann(M_P) = (ann(M))_P = 0_P = 0$, so that $M_P$ is a faithful $R_P$ −module and hence we get $M_P$ is a faithful multiplication $R_P$ −module. Since, $S_M(PM) \subseteq P$ and $PM$ is prime, by **Proposition 3.15**, we get $(PM)_P$ is a prime submodule of $M_P$, that is $P_P M_P$ is a prime submodule of $M_P$ and since, $P_P$ is a prime ideal of $R_P$, ($P_P$ is the unique maximal ideal of the local ring $R_P$), so by **[13, Corollary 2.4]**, we get $(P_P M_P : M_P) = P_P$, that is $((PM)_P : M_P) = P_P$. As, $S_M(PM) \subseteq P$, by **Theorem 3.4**, we get $((PM)_P : M_P) = (PM : M)_P$, so that $(PM : M)_P = P_P$. Now, let $x \in (PM : M)$, then we get $\frac{x}{1} \in (PM : M)_P = P_P$, so that $px \in P$ for some $p \notin P$ and as $P$ is prime, we get $x \in P$. Hence, $(PM : M) \subseteq P$. Next we will show that $S_R(PM : M) \subseteq S_M(PM)$. Let $r \in S_R(PM : M)$, then $rs \in PM : M$ for some $s \notin (PM : M)$, that gives $rsM \subseteq PM$ and $sM \nsubseteq PM$, so that there exists $m \in M$ such that $sm \notin PM$, but then $rsm \in PM$. Thus $r \in S_M(PM)$. Hence $S_R(PM : M) \subseteq S_M(PM) \subseteq P$. Finally we will show that $P \subseteq (P : M)$. Let $x \in P$, then we get $\frac{x}{1} \in P_P = (PM : M)_P$, so that $qx \in (PM : M)$ for some $q \notin P$. If $x \notin (PM : M)$, then we get $q \in$

$S_R(PM:M) \subseteq S_M(PM) \subseteq P$ which is a contradiction, so that we must have $x \in (PM:M)$. Hence, $P \subseteq (PM:M)$ and thus we get $(PM:M) = P$.
Conversely, suppose that $(PM:M) = P$. If $PM = M$, then $(M:M) = P$ and as $1 \notin P$, we get $1 \notin (M:M)$. This gives that $1.M \nsubseteq M$, that is $M \nsubseteq M$, is a contradiction, so that $PM \neq M$, that is $PM$ is a proper submodule of $M$. Next, let $rx \in PM$, where $r \in R$ and $x \in M$. If $x \notin PM$, then $r \in S_M(PM) \subseteq P$, that means, $r \in P$. Hence, $r \in (PM:M)$, that gives $rM \subseteq PM$. Thus $PM$ is a prime submodule of $M$.

**Proposition 4.5**. Let $M$ be a locally multiplication $R$ −module and $P$ a prime ideal of $R$ such that $S_M(PM) \subseteq P$. If $PM$ is a maximal submodule of $M$, then $(PM:M) = P$.

**Proof**. As $P$ is a prime ideal of $R$, we have $M_P$ is a multiplication $R_P$ −module. As $PM$ is a maximal submodule of $M$ and $S_M(PM) \subseteq P$, by **Proposition 3.16**, we get $(PM)_P$ is a maximal submodule of $M_P$, that is, $P_P M_P$ is a maximal submodule of the multiplication $R_P$ −module $M_P$. Since, $P_P$ is a maximal ideal of $R_P$, (in fact, $P_P$ is the unique maximal ideal of $R_P$ since $R_P$ is a local ring), so that by **[13, Corollary 2.5]**, we get $(P_P M_P : M_P) = P_P$, which gives $((PM)_P : M_P) = P_P$. Since, $PM$ is a maximal submodule of $M$, so it is proper in $M$ and as $S_M(PM) \subseteq P$, by **Theorem 3.4**, we get $((PM)_P : M_P) = (PM:M)_P$, so that we get $(PM:M)_P = P_P$. Since $S_R(P) = P \subseteq P$, so by **[7, Lemma 4.8]**, we get $(PM:M) \subseteq P$. Conversely, let $x \in P$. Clearly we have $xM \subseteq PM$, so that $x \in (PM:M)$ and thus $P \subseteq (PM:M)$. Hence, we get $(PM:M) = P$.

Next, we prove that a faithful locally multiplication are finitely generated under certain conditions.

**Proposition 4.6**. If $M$ is a faithful locally multiplication $R$ −module with $S_M(0) \subseteq Rad(R)$, then the following statements hold.
**(1)** If $S_M(N) \subseteq Rad(R)$, for every finitely generated submodule $N$ of $M$, then $M$ is finitely generated.
**(2)** $AM \neq M$ for any proper ideal $A$ of $R$ with $S_R(A) \subseteq Rad(R)$.

**Proof**. As $R$ is commutative with identity, it contains at least one maximal ideal say $P$ which is also prime and so that $M_P$ is a multiplication $R_P$ −module. Since, $S_M(0) \subseteq Rad(R) \subseteq P$, so by **Proposition 3.10**, we get $ann(M_P) = (ann(M))_P$. When $M$ is faithful, we get $ann(M) = 0$, and hence, $ann(M_P) = (ann(M))_P = 0_P = 0$. Thus $M_P$ is a faithful multiplication $R_P$ −module.
**(1)** By using **[13, Theorem 2.6]**, we get $M_P$ is finitely generated, so let $M_P = R_P \frac{m_1}{p_1} + R_P \frac{m_2}{p_2} + \cdots + R_P \frac{m_k}{p_k}$, where $m_i \in M$ and $p_i \notin P$ for $i$ $(1 \le i \le k)$. By **[9, Corollary 2.9]**, we have $M_P = (Rm_1)_P + (Rm_2)_P + \cdots + (Rm_k)_P = (Rm_1 + Rm_2 + \cdots + Rm_k)_P$. Since, $Rm_1 + Rm_2 + \cdots + Rm_k$ is a finitely generated submodule of $M$ with the generator set $\{m_1, m_2, \dots, m_k\}$, so that $S_M(Rm_1 + Rm_2 + \cdots + Rm_k) \subseteq Rad(R) \subseteq P$ and $M_P \subseteq (Rm_1)_P + (Rm_2)_P + \cdots + (Rm_k)_P$. Then by **[7, Lemma 4.8]**, we get $M \subseteq Rm_1 + Rm_2 + \cdots + Rm_k$, from which we get that $M = Rm_1 + Rm_2 + \cdots + Rm_k$. Hence, $M$ is finitely generated.
**(2)** Suppose $AM = M$, for some proper ideal $A$ of $R$ with $S_R(A) \subseteq Rad(R)$. Then $A_P$ is an ideal of $R_P$ and $S_R(A) \subseteq Rad(R) \subseteq P$. Now, we have $A_P M_P = (AM)_P = M_P$. Since $M_P$ is a faithful multiplication $R_P$ −module, the ideal $A_P$ is not proper in $R_P$. Therefore $A_P = R_P$. Clearly $R_P \subseteq A_P$, so by **Lemma 3.8**, we get $A = R$, which is a contradiction. Hence, $AM \neq M$ for any proper ideal $A$ of $R$ with $S_R(A) \subseteq Rad(R)$.

Now we give some conditions which make a submodule of a faithful locally multiplication module a product of a largest ideal in $R$ by the module.

**Proposition 4.7**. Let $M$ be a faithful locally multiplication $R$ −module and $S_M(IM) \subseteq Rad(R)$ for every ideal $I$ of $R$. If $N$ is a submodule of $M$ such that $S_M(N) \subseteq Rad(R)$, then there exists a largest ideal $A$ of $R$, (in the sense of inclusion), such that $N = AM$.

**Proof**. Let $N$ be a submodule of $M$. As $R$ is commutative with identity, it contains at least one maximal ideal say, $P$ which is also prime. Thus $M_P$ is a multiplication $R_P$ −module. Since, $S_M(IM) \subseteq Rad(R)$ for every ideal $I$ of $R$, so by taking $I = 0$, we get that $S_M(0) = S_M(0M) \subseteq$

$Rad(R) \subseteq P$. By **Proposition 3.10**, we get $ann(M_P) = (ann(M))_P$. As, $M$ is faithful, we get $ann(M) = 0$, and hence, $ann(M_P) = 0$. Thus $M_P$ is a faithful multiplication $R_P$ −module. Since $N_P$ is a submodule of $M_P$, by **[13, Theorem 2.8]**, there exists a unique ideal $\overline{A}$ of $R_P$ such that $N_P = \overline{A}M_P$. From **Corollary 3.18**, there exists a unique ideal $A$ of $R$ with the properties $S_R(A) \subseteq P$ and $\overline{A} = A_P$. Thus $N_P = A_P M_P = (AM)_P$ and since we have $S_M(N) \subseteq Rad(R) \subseteq P$ and $S_M(AM) \subseteq Rad(R) \subseteq P$, so by **[7, Lemma 4.8]**, we get $N = AM$. If $B$ is any other ideal of $R$ such that $N = BM$, then $A_P M_P = N_P = (BM)_P = B_P M_P$, and by the uniqueness of the ideal $\overline{A}$ of $R_P$ for which $N_P = \overline{A}M_P$, we get $B_P = \overline{A} = A_P$. Hence $B_P \subseteq A_P$. Since $S_R(A) \subseteq P$, by **Lemma 3.8**, we get $A$ is a maximal ideal of $R$.

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